Is the brownian bridge a good noise model on the circle?

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Abstract: The aim of this article is to analyze stationary, periodic and gaussian processes in order to provide a simple and general method to construct noise models on the circle of arbitrary regularity. This method allows to build models with feasible computational parameter estimates. Properties of asymptotic ML estimates are provided together with results on path regularity of such processes. Moreover, we show that the brownian bridge cannot be a good noise model on the circle.

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1. Introduction

The brownian bridge is a universally known model which is widely used in several areas of applied mathematical science. As only an example, the recent publication of [8, 10] and the reference therein provide a huge interested literature, while [2] provide a theoretical analysis of such a process. The main aspect of the browinan bridge is its periodicity, that makes this model widely used. On the other hand, a lack of this model is its non-stationarity, which is almost a must when one models pure noise. Motivated by these two facts, we focus here on $\mathscr{H}$, the set of gaussian, stationary, periodic processes, which can be thought as the space of the 'noises' on a periodic data, e.g. the noise on a contour of a closed feature on an image, or, more generally, the noise on a stationary, periodic, functional data process. In this paper, the set $\mathscr{H}$ is studied in order to provide a simple and general method for constructing models of arbitrary regularity, which can be used in modeling real data. A "good" noise model is then generated by a process in $\mathscr{H}$ conditioned to be 0 when $t = 0$. Surprisingly, the brownian bridge can not be generated in this way. Indeed, it is generated by a gaussian, stationary, *antiperiodic* processes conditioned to be 0 when $t = 0$ (and hence periodic!).

The paper is organized as follows. In the next Section 2, we recall that $\mathscr{H}$ is isometrically equivalent to ℓ^2 as a consequence of Karhunen-Loève's theorem.

Notably, the isometry is build via Fourier transform. In particular, FFT is used in application, which makes the set $\mathscr{H}$ computationally feasible.

In Section 3, we show that the brownian bridge can not be obtained by conditioning a process in $\mathscr{H}$. In Section 4 we show that any process in $\mathscr{H}$ shares asymptotically the spectrum with itself when it is "conditioned to be 0 at $t = 0$". In Section 5 it is proven that also the path regularity is maintained for such couples.

The Section 6 presents a simple parametric model in $\mathscr{H}$ for constructing noise models on the circle with path processes of arbitrary regularity. Finally, maximum likelihood asymptotic properties are studied for the parameters of this model.

For what concerns notations, $s, t, \ldots$ relates to time variables, and will often belong to $[0, 1]$. We denote by $\{x_t\}_{t\in[0,1]}, \{y_t\}_{t\in[0,1]}, \ldots$ stochastic adapted process defined on a given filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,1]}, \mathbb{P})$, while $(X_n)_n$, $(Y_n)_n$, $(Z_n)_n, \ldots$ are sequences of random variables. $C(s, t)$ is a positive semidefinite function (it will be the correlation function of a stochastic process). When a process is stationary, its covariance function will often be replaced by the associated covariogram function $\tilde{C}(t-s) = C(s, t)$. The sequence $(e_k(t))_{k\in\mathbb{N}}$ denotes a sequence of orthogonal function on $L^2([0, 1])$. Finally, we denote by $[\![t]\!]_1$ the fractional part sawtooth function of the real number t, which is defined by the formula $[\![t]\!]_1 = t - \text{floor}(t)$.

2. Preliminaries and Karhunen-Loéve's decomposition theorem

In this section we recall some basic results from gaussian processes theory. The first theorem we need is the Karhunen-Loéve's decomposition theorem (see [7]), that states what follows.

Theorem 2.1 (Karhunen-Loéve). *Let $\{x_t\}_{t\in[0,1]}$, such that $E[x_t] \equiv 0$, and $Cov(x_t, x_s) = C(t, s)$, continuous in both variables. Then $x_t = \sum_{k=1}^{\infty} Z_k\, e_k(t)$, where*

- *the functions $(e_k(\cdot))_k$ are the eigenfunctions of the following integral operator from $L^2[0, 1]$ in itself*

$$f \in L^2[0, 1] \longrightarrow g(t) = \int_0^1 C(t, \tau) f(\tau) d\tau, \tag{2.1}$$

and $(e_k\cdot))_k$ form an orthonormal basis for the space spanned by the eigenfunctions corresponding to nonzero eigenvalues;

- *the random variables $Z_1, Z_2, \ldots$ are given by $Z_k = \int_0^1 x_t e_k(t) dt$ and form a zero-mean orthogonal system (i.e., $E(Z_k Z_j) = 0$ for $k \neq j$) with variance λ_k^2, where λ_k^2 is the eigenvalue corresponding to the eigenfunction $e_k(\cdot)$.*

The series $\sum_{k=1}^{\infty} Z_k e_k(t)$ converges in mean square to x_t, uniformly in t:

$$\sup_{t\in[0,1]} E\Big((x_t - \sum_{k=1}^{\infty} Z_k e_k(t))^2\Big) \underset{n\to\infty}{\longrightarrow} 0.$$

Finally, x_t is a gaussian process if and only if $(Z_k)_k$ is a sequence of independent gaussian random variables.

2.1. Representation of the set $\mathscr{H}$ with respect to the Fourier basis $(\mathbf{s}_k(t), \mathbf{c}_k(t))_k$

We deal in this paper with the following set $\mathscr{H}$ of processes, thought as the set of 'pure gaussian noises' on the unit circle.

Definition 2.1. Let $\{x_t\}_{t\in[0,1]}$ be a stochastic process with covariance function $C(s,t) = Cov(x_t, x_s)$. $\mathscr{H}$ is the set of real *gaussian* stochastic processes $\{x_t\}_{t\in[0,1]}$ such that

zero-mean: $E(x_t) = 0$, $\forall t \in \mathbb{R}$;
continuously stationary: there exists a continuous real function $\tilde{C}$ such that $C(s,t) = \tilde{C}(s-t)$, $\forall s,t \in \mathbb{R}$;
periodic: $\{x_t\}_{t\in[0,1]}$ admits a periodical version (i.e. $x_0 = x_1, a.s.$).

Remark 1. A necessary and sufficient condition for a continuously stationary process to be periodic is that $\tilde{C}(1) = \tilde{C}(0)$. This allows a continuous version of the process with $Var(x_{t+1} - x_t) = 0$ for any $t \in \mathbb{R}$.

Remark 2. We remark that if $\{x_t\}_{t\in[0,1]} \in \mathscr{H}$ and if $\tilde{C}(s-t) = C(s,t)$ is its covariogram function, then $\tilde{C}(t) = \tilde{C}(t+1)$.

The set $\mathscr{H}$ is a Hilbert space, when it is equipped with the inner product given by

$$(\{x_t\}_{t\in[0,1]}, \{y_t\}_{t\in[0,1]}) = \int_0^1 E(x_t y_t) dt \in \mathbb{R}_+.$$

Karhunen-Loéve's decomposition theorem can be specialized to $\mathscr{H}$, in order to show that a process is in $\mathscr{H}$ if and only if it can be written as limit of a canonical trigonometric random series, namely the sequence $1, (\mathbf{s}_k(t), \mathbf{c}_k(t))_k$, where $\mathbf{s}_k(t) = \sqrt{2}\sin(2k\pi t)$ and $\mathbf{c}_k(t) = \sqrt{2}\cos(2k\pi t)$.

Theorem 2.2. *Let $\{x_t\}_{t\in[0,1]} \in \mathscr{H}$ with covariance $C(s,t) = \tilde{C}(t-s)$; then in mean square, uniformly in t,*

$$x_t = c_0 Y_0' + \sum_{k=1}^{\infty} c_k (Y_k \mathbf{s}_k(t) + Y_k' \mathbf{c}_k(t))$$

where $(Y_n)_n, (Y_n')_n$ are two independent sequence of independent standard gaussian variables, and $(c_k)_k \in \ell^2$ is such that

$$c_n^2 = \int_0^1 \tilde{C}(s)\cos(2n\pi s) ds, \qquad n = 0, 1, 2, \ldots$$

Proof. See Appendix A. □

Theorem 2.3. *Let $(Y_n)_n, (Y'_n)_n$ be two independent sequence of independent standard gaussian variables, and $(c_k)_k \in \ell^2$. Then the sequence*

$$y_t^{(n)} = c_0 Y'_0 + \sum_{k=1}^{n} c_k (Y_k \mathbf{s}_k(t) + Y'_k \mathbf{c}_k(t))$$

converges in mean square, uniformly in t to $\{y_t\}_{t\in[0,1]} \in \mathscr{H}$. Moreover if $C(s,t)$ is the y_t covariance function, then uniformly, absolutely and in $L^2[0,1]\times[0,1]$,

$$\begin{aligned} C(s,t) &= c_0^2 + \sum_{k=1}^{\infty} c_k^2 \mathbf{c}_k(s)\mathbf{c}_k(t) + \sum_{k=1}^{\infty} c_k^2 \mathbf{s}_k(s)\mathbf{s}_k(t) \\ &= c_0^2 + 2\sum_{k=1}^{\infty} c_k^2 \cos(2k\pi s)\cos(2k\pi t) + 2\sum_{k=1}^{\infty} c_k^2 \sin(2k\pi s)\sin(2k\pi t) \qquad (2.2) \\ &= c_0^2 + 2\sum_{k=1}^{\infty} c_k^2 \cos(2k\pi(s-t)). \end{aligned}$$

Proof. See Appendix A. □

Remark 3. As a consequence of the Theorem 2.2 and Theorem 2.3, we can observe that periodic processes with period $\frac{1}{m}$ have only mk terms in their expansion:

$$x_{t+\frac{1}{m}} = c_0 Y'_0 + \sum_{k=1}^{\infty} c_{mk}(Y_{mk}\mathbf{s}_{mk}(t+\tfrac{1}{m}) + Y'_{mk}\mathbf{c}_{mk}(t+\tfrac{1}{m})) = x_t$$

More fancy, processes having only odd terms are antiperiodic with period $\frac{1}{2}$, i.e.

$$x_{t+\frac{1}{2}} = \sum_{k=0}^{\infty} c_{2k+1}(Y_{2k+1}\mathbf{s}_{2k+1}(t+\tfrac{1}{2}) + Y'_{2k+1}\mathbf{c}_{2k+1}(t+\tfrac{1}{2})) = -x_t.$$

An immediate consequence of this remark is that when one needs to model a pure noise on the circle, then he must choose processes whose expansion have both odd and even terms.

2.2. *The quotient set $\mathscr{H}_Z$*

It is easy to see that $\mathscr{H}$ can be seen as a Hilbert space, isometrically equivalent to the space of the coefficients ℓ^2; let us consider a couple $Z = ((\bar{Y}_n)_n, (\bar{Y}'_n)_n)$ of independent sequence of independent standard gaussian variables. For each $\{z_t\}_{t\in[0,1]} \in \mathscr{H}$, there exists an $\{x_t\}_{t\in[0,1]} \in \mathscr{H}_Z$ having the same law, where

$$\mathscr{H}_Z = \Big\{ \{x_t\} \in \mathscr{H} : x_t = a_0 \bar{Y}'_0 + \sum_{k=1}^{\infty} a_k(\bar{Y}_k \mathbf{s}_k(t) + \bar{Y}'_k \mathbf{c}_k(t)), (a_n)_n \in \ell^2 \Big\}$$

and the limit is in mean square and uniformly in t. From Theorem 2.2 and Theorem 2.3 it is naturally defined an isometry between the representative space $\mathscr{H}_Z$ and ℓ^2:

$$x_t = a_0 \bar{Y}'_0 + \sum_{k=1}^{\infty} a_k(\bar{Y}_k \mathbf{s}_k(t) + \bar{Y}'_k \mathbf{c}_k(t)) \longleftrightarrow (a_0, \sqrt{2}a_1, \sqrt{2}a_2, \sqrt{2}a_3, \ldots)_n \in \ell^2,$$

where $\|x_t\|_{\mathscr{H}_Z} = \sqrt{a_0^2 + 2\sum_n a_n^2}$.

2.3. The space $\mathscr{H}_0$ and its relation with $\mathscr{H}_Z$

By Theorem 2.3 given $(c_i)_i \in \ell^2$, there exists a unique $\{x_t, t \in [0,1]\} \in \mathscr{H}_Z$, with covariance function given by

$$C(s,t) = c_0^2 + \sum_{k=1}^{\infty} c_k^2 \mathbf{c}_k(s)\mathbf{c}_k(t).$$

The process $\{x_t\}$, conditioned to be 0 at $t = 0$, is the periodic zero-mean gaussian process $\{y_t\}$ with covariance function

$$R(s,t) = C(s,t) - \frac{C(s,0)C(0,t)}{C(0,0)}. \tag{2.3}$$

Let us define the set $\mathscr{H}_0$ of such processes.

Definition 2.2. Let $\mathscr{H}_0$ be the following set

$$\begin{aligned}\mathscr{H}_0 = \{\{y_t\}_{t\in[0,1]} \colon \exists \{x_t\}_{t\in[0,1]} \in \mathscr{H} \text{ such that} \\ \mathscr{L}((y_{t_1}, \ldots, y_{t_n})) = \mathscr{L}((x_{t_1}, \ldots, x_{t_n})|x_0 = 0), \quad \forall \underline{t} \in [0,1]^n, n \in \mathbb{N}\}.\end{aligned}$$

We call:

Generator process: the process $\{x_t\}_{t\in[0,1]} \in \mathscr{H}$;
Generated process: the process $\{y_t\}_{t\in[0,1]} \in \mathscr{H}_0$.

It is easy to show that $\{y_t\}_{t\in[0,1]} \notin \mathscr{H}$ because it is not stationary. However, the function $R(s,t)$ is symmetric, and hence it is the L_2-limit of its 2-D Fourier series. With the notation given above, with $\mathbf{c}_0(t) = 1$, we get the series expansion:

$$\begin{aligned}R(s,t) = &+ \sum_{k,j=0}^{\infty} r_{kj}^{cc} \mathbf{c}_k(s)\mathbf{c}_j(t) + \sum_{k,j=1}^{\infty} r_{kj}^{ss} \mathbf{s}_k(s)\mathbf{s}_j(t) \\ &+ \sum_{k=1,j=0}^{\infty} r_{kj}^{sc} \mathbf{s}_k(s)\mathbf{c}_j(t) + \sum_{k=0,j=1}^{\infty} r_{kj}^{cs} \mathbf{c}_k(s)\mathbf{s}_j(t).\end{aligned} \tag{2.4}$$

The following theorem gives a necessary and sufficient condition for a process $\{y_t, t \in [0,1]\}$ with covariance function $R(s,t)$ to have a unique process $\{x_t\} \in \mathscr{H}_Z$ which generates it. The trivial case when $R(s,t) = 0$ (generated by a constant process) is omitted since it is the sole case when the solution is not unique. The proof may be found in Appendix A.

Theorem 2.4. *For any gaussian process $\{y_t, t \in [0,1]\}$ such that $y_0 = 0$, $E(y_t) = 0$ and continuous covariance function $R(s,t) \neq 0$, there exists a unique (in law) stationary process $\{x_t\} \in \mathscr{H}_Z$ which generates $\{y_t\}$ if and only if the Fourier coefficients of $R(s,t)$ satisfy:*

- *the mixed matrices* $\cos - \sin$ *and* $\sin - \cos$ *are null:*

$$(r^{cs}_{jk})_{j\geq 0, k\geq 1} = (r^{sc}_{jk})_{j\geq 1, k\geq 0} = 0;$$

- *the* $\sin - \sin$ *matrix is a non-negative diagonal in ℓ^1:*

$$(r^{ss}_{jk})_{j,k\geq 1} = \begin{pmatrix} r^{ss}_{11} & 0 & 0 & 0 & \cdots \\ 0 & r^{ss}_{22} & 0 & 0 & \cdots \\ 0 & 0 & r^{ss}_{33} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

  with $r^{ss}_{kk} \geq 0$ and $r^{cc}_{00} < \bar{r} = \sum_k r^{ss}_{kk} < \infty$;
- *defined $r^{ss}_{00} = \frac{r^{cc}_{00}\bar{r}}{\bar{r} - r^{cc}_{00}}$, the* $\cos - \cos$ *matrix is built from the* $\sin - \sin$ *matrix and r^{cc}_{00}:*

$$(r^{cc}_{jk})_{j,k\geq 0} = \begin{pmatrix} r^{ss}_{00} & 0 & 0 & 0 & \cdots \\ 0 & r^{ss}_{11} & 0 & 0 & \cdots \\ 0 & 0 & r^{ss}_{22} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} - \frac{\bar{r} - r^{cc}_{00}}{\bar{r}^2} \begin{pmatrix} r^{ss}_{00}r^{ss}_{00} & r^{ss}_{00}r^{ss}_{11} & r^{ss}_{00}r^{ss}_{22} & \cdots \\ r^{ss}_{11}r^{ss}_{00} & r^{ss}_{11}r^{ss}_{11} & r^{ss}_{11}r^{ss}_{22} & \cdots \\ r^{ss}_{22}r^{ss}_{00} & r^{ss}_{22}r^{ss}_{11} & r^{ss}_{22}r^{ss}_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

3. On the brownian bridge

In the previous sections we built the set $\mathscr{H}$ as the set of the gaussian stationary and periodical processes. A process in this set should be used when one models a pure gaussian noise on the circle. When one needs the process to be zero at the starting point, he should restrict the choice to $\mathscr{H}_0$.

One of the models mainly used for periodic noise is the brownian bridge, i.e. the process $\{y_t\}_{t\in[0,1]}$ such that $y_t = W_t - tW_1$, where W_t is a brownian motion. This process is gaussian, periodic and not stationary, since $y_0 = y_1 \equiv 0$. In order to be a good model for pure noise the brownian bridge should belong to $\mathscr{H}_0$, i.e., it should be the conditioned process of a stationary process $\{x_t\} \in \mathscr{H}$. The next theorem shows that this is not the case.

Theorem 3.1. *Let $\{y_t\}_{t\in[0,1]}$ be a brownian bridge. There does not exists a process $\{x_t\} \in \mathscr{H}$ which generates $\{y_t\}_{t\in[0,1]}$, i.e. $\{y_t\}_{t\in[0,1]} \notin \mathscr{H}_0$.*

Proof. See Appendix B. □

The Theorem 3.1 states that no process in $\mathscr{H}$ generates a brownian bridge. The following question is if the paths of a brownian bridge may be extended on $[0,2]$ in such a way that the extended process is in $\mathscr{H}_0(0,2)$. The answer is quite surprising, if we do not underling the following result.

Theorem 3.2. *Let* $\{z_t\}_{t\in[0,1]}$ *be a non deterministic, gaussian process with* $z_0 = z_1 = 0$ *and continuous covariance function* $R(s,t)$, $s,t \in [0,1]$. *There is at most one extension of* $\{z_t\}_{t\in[0,1]}$ *to* $\{\tilde{z}_t, t \in [0,2]\} \in \mathscr{H}_0(0,2)$. *Moreover,* $\tilde{z}_t$ *is either* 1*–antiperiodic or* 1*–periodic (see Remark 3).*

Proof. See Appendix B. □

The brownian bridge is generated by an 1–antiperiodic stationary process.

Theorem 3.3. *There exists a unique extension* $\{\tilde{y}_t, t \in [0,2]\} \in \mathscr{H}_0(0,2)$ *of the brownian bridge* $\{y_t\}_{t\in[0,1]}$. *Moreover,* $\{\tilde{y}_t, t \in [0,2]\}$ *is generated by*

$$\tilde{x}_t = \sum_{k=1}^{\infty} \frac{2}{\pi(2k+1)} \big(Y_k \sin(\pi(2k+1)t) + Y'_k \cos(\pi(2k+1)t) \big), \tag{3.1}$$

which is 1*–antiperiodic.*

Proof. See Appendix B. □

4. A process in $\mathscr{H}_0$ shares the same asymptotic behavior for the spectrum with its generator

We want to get information about Fourier coefficients of Karhunen-Loéve expansion for processes in $\mathscr{H}_0$ with respect to the coefficients of their generators in $\mathscr{H}$. To do this, as described in the Theorem 2.1, it is sufficient to study the spectrum of the integral operator induced by the covariance function of the process $\{y_t\}_{t\in[0,1]} \in \mathscr{H}_0$ generated by $\{x_t\}_{t\in[0,1]} \in \mathscr{H}$.

Theorem 4.1. *Denote by* $\{y_t\}_{t\in[0,1]}$ *a process in* $\mathscr{H}_0$ *and by* $\{x_t\}_{t\in[0,1]}$ *its generator in* $\mathscr{H}$. *Let* $\{c_n\}_{n\in\mathbb{N}} \in \ell^2$ *be the sequence of Fourier coefficients of Karhunen-Loéve expansion of the process* $\{x_t\}_{t\in[0,1]}$ *(as in Theorem 2.2 and Theorem 2.3), and let* $\{\tilde{c}_n\}_{n\in\mathbb{N}}$ *be the sequence of Fourier coefficients of Karhunen-Loéve expansion of the process* $\{y_t\}_{t\in[0,1]}$. *Then the sequences are asymptotic:* $(\tilde{c}_n)_n \asymp (c_n)_n$.

Proof. See Appendix C. □

5. A process in $\mathscr{H}_0$ shares the same path regularity properties with its generator.

We showed in Theorem 4.1 that a process in $\mathscr{H}_0$ and its generator in $\mathscr{H}$ share the same asymptotic behavior for the spectrum. In this section, we show that the regularity of the paths is also maintained.

5.1. Holder regularity of the paths of processes in $\mathscr{H}$ and in $\mathscr{H}_0$

We first remind that the Holder space $C^{k,\alpha}([0,1])$, where $k \geq 0$ is an integer and $0 < \alpha \leq 1$, consists of those functions on $[0,1]$ having continuous derivatives up to order k and such that the k^{th}-derivative is Holder continuous with exponent α. We recall a classic regularity theorem.

Theorem 5.1 (see [12]). *Let $\{x_t\}_{t\in[0,1]}$ a real stochastic process such that there exist three positive constants γ, c and ϵ so that*

$$E\big(|x_t - x_s|^\gamma\big) \leq c|t-s|^{1+\epsilon};$$

then there exists a modification $\{\tilde{x}_t\}_{t\in[0,1]}$ of $\{x_t\}_{t\in[0,1]}$, such that

$$E\Big(\Big(\sup_{s\neq t}\frac{|\tilde{x}_t - \tilde{x}_s|}{|t-s|^\alpha}\Big)^\gamma\Big) < \infty$$

for all $\alpha \in [0, \frac{\epsilon}{\gamma})$; in particular the trajectories of $\{\tilde{x}_t\}_{t\in[0,1]}$ belongs to $C^{0,\alpha}([0,1])$.

The following results are a immediate consequence of this last theorem (proofs may be found in Appendix D), where the processes $\{\tilde{x}_t\}_{t\in[0,1]}$ and $\{\tilde{y}_t\}_{t\in[0,1]}$ are thought modified as in the Theorem 5.1.

Theorem 5.2. *Let $\{\tilde{x}_t\}_{t\in[0,1]}$, a stochastic stationary process with null expectation, and let $R(s,t)$ be its covariance function; if $R \in C^{0,\alpha}([0,1]\times[0,1])$ then almost all trajectories of $\{\tilde{x}_t\}_{t\in[0,1]}$ belong to $C^{0,\beta}([0,1])$ with $\beta < \frac{\alpha}{2}$.*

It is simple to apply this last theorem to processes laying in $\mathscr{H}$ and in $\mathscr{H}_0$: assume that $\{x_t\}_{t\in[0,1]} \in \mathscr{H}$ and let $C(s,t) = \tilde{C}(s-t)$ be its covariance function. If $\tilde{C} \in C^{0,\alpha}([0,1])$, then almost all trajectories of $\{\tilde{x}_t\}_{t\in[0,1]}$ belongs to $C^{0,\beta}([0,1])$, for any $\beta < \frac{\alpha}{2}$. The same argument can be applied to $\mathscr{H}_0$ processes.

In fact we can say something more.

Theorem 5.3. *Let $\{x_t\}_{t\in[0,1]} \in \mathscr{H}$ and let $C(s,t) = \tilde{C}(s-t)$ be its covariance function. Consider its generated $\mathscr{H}_0$ process $\{y_t\}_{t\in[0,1]} \in \mathscr{H}_0$, and let $R(s,t)$ be its covariance function. Then we have*

$$\tilde{C} \in C^{0,\alpha}([0,1]) \Rightarrow R \in C^{0,\alpha}([0,1]\times[0,1]) \Rightarrow \begin{cases}\{\tilde{y}_t\}_{t\in[0,1]} \in C^{0,\beta}([0,1])\\ \{\tilde{x}_t\}_{t\in[0,1]} \in C^{0,\beta}([0,1])\end{cases}$$

This last result implies that regularity properties of almost all trajectories of $\{\tilde{x}_t\}_{t\in[0,1]}$ and of its generated process $\{\tilde{y}_t\}_{t\in[0,1]}$ have the same lower bound, obtained by studying regularity of their covariance function.

5.2. *Upper order regularity*

In Section 2.2, a sequence in ℓ^2 is uniquely associated to each stochastic process in $\mathscr{H}$. We are now showing how the decrease rate of such sequence ia sasociated with the regularity of the process trajectory path.

A very useful result for our analysis will be the following one, whose proof may be found in [9].

Theorem 5.4 (Boas' Theorem)**.** *Let $f \in L^1[0,1]$ be a function whose Fourier expansion has only nonnegative cosine terms, and let $(a_n)_n$ be the sequence of its cosine coefficient. Then*

$$f \in C^{0,\alpha}([0,1]) \Longleftrightarrow a_k = O\Big(\frac{1}{k^{\alpha+1}}\Big).$$

Boas' Theorem may be used in connection with Theorem 2.2 and Theorem 2.3 to deduce more regularity properties of the processes in $\mathscr{H}$, since $\tilde{C}$ is a function whose Fourier expansion has only nonnegative cosine terms. In fact, take $(c_n)_n$ as in Theorem 2.2 and Theorem 2.3. From Boas' Theorem we have that if $k^2c_k^2 = O(\frac{1}{k^{1+\alpha}})$ for $0 < \alpha \leq 1$, then $\tilde{C} \in C^{2,\alpha}([0,1])$. This link between the regularity of $\tilde{C}$ and the paths of $\{x_t\}_{t\in[0,1]}$ is underlined in the following theorem. The proof is in the Appendix D.

Theorem 5.5. *With the notations of Theorem 2.3, if $c_k^2 = O(\frac{1}{k^{3+\alpha}})$, then there exists a version of $\{x_t\}_{t\in[0,1]}$ whose trajectories belongs to $C^{1,\beta}([0,1])$, with $\beta < \frac{\alpha}{2}$.*

A natural generalization of this result is the following corollary.

Corollary 5.1. *With the notations of Theorem 2.3, if $c_k^2 = O(\frac{1}{k^{1+2m+\alpha}})$ then there exists a version of $\{x_t\}_{t\in[0,1]}$whose trajectories belongs to $C^{m,\beta}([0,1])$, with $\beta < \frac{\alpha}{2}$.*

6. A parametric model in $\mathscr{H}$ generalizing the brownian bridge

Results provided in this paper allow to create a gaussian parametric family of stationary and periodic processes of arbitrary regularity. In fact, let us consider the following family of processes in $\mathscr{H}$:

$$x_t = \sum_{k=1}^{\infty} \frac{a}{k^p}(Y_k \sin(2k\pi t) + Y'_k \cos(2k\pi t)). \tag{6.1}$$

Theorem 5.2 states that the paths become more regular as p increases. This property is shown in Figure 1, which suggests how to smooth a process by changing p.

Model (6.1) gives a family of gaussian processes whose trajectories are arbitrarily regular. In application, maximum likelihood estimates of a and p is a straightforward consequence of a FFT of the observed discretized process $\{x_t\}_{t\in[0,1]}$. The property of these estimators are studied in the following section.

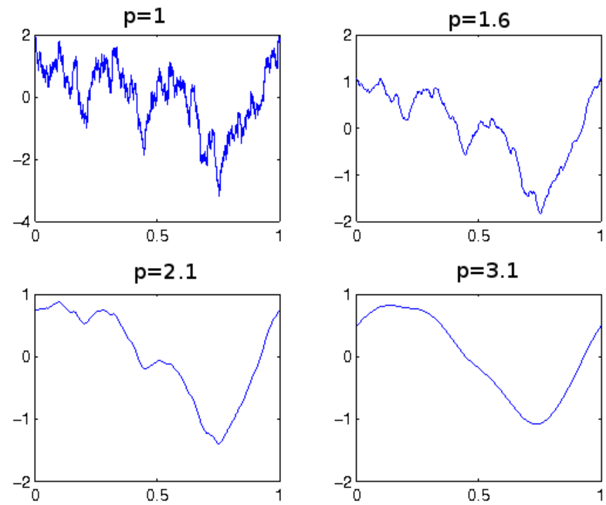

FIG 1. *Change of path regularity: a comparison between trajectories of process x_t given in* (6.1) *for fixed* $(Y_k, Y'_k)_k$ *and varying the value of* p.

6.1. Maximum likelihood estimators of (6.1)

Given $(x_{t_0}, x_{t_1}, \dots, x_{t_n})$ sampled from (6.1), we want to find the property of the maximum likelihood estimator $(\hat{a}, \hat{p})$ of the parameters (a, p).

More precisely, with a equispaced or nonequispaced Fourier transform (see, e.g., [3, 4]), we first transform $(x_{t_0}, x_{t_1}, \dots, x_{t_n})$ into $(y_1^{(1)}, y_2^{(1)}, \dots, y_n^{(1)})$ and $(y_1^{(2)}, y_2^{(2)}, \dots, y_n^{(2)})$ (real and imaginary part). As a consequence of Theorem 2.2 applied to (6.1), there exist two sequences $(Y_k)_k$ and $(Y'_k)_k$ of independent gaussian standard random variables such that

$$\begin{array}{llll} y_1^{(1)} = aY_1, & y_2^{(1)} = \frac{a}{2^p} Y_2, & \dots, & y_n^{(1)} = \frac{a}{n^p} Y_n, \\ y_1^{(2)} = aY'_1, & y_2^{(2)} = \frac{a}{2^p} Y'_2, & \dots, & y_n^{(2)} = \frac{a}{n^p} Y'_n. \end{array}$$

The log-likelihood function then reads

$$\begin{aligned} \ell_n(a,p) = -n\log(2\pi) - 2n\log(a) + 2p\sum_{k=1}^n \log(k) \\ - \frac{1}{2a^2}\sum_{k=1}^n k^{2p}\big((y_k^{(1)})^2 + (y_k^{(2)})^2\big) \end{aligned}$$

and hence, if $o_k = (y_k^{(1)})^2 + (y_k^{(2)})^2$, $k = 1, \dots, n$ and $\psi = \sum_{k=1}^n \log(k)$, we get

$$\begin{aligned} \frac{\partial \ell_n}{\partial a} &= -\frac{2n}{a} + \frac{1}{a^3} \sum_{k=1}^n k^{2p} o_k \\ \frac{\partial \ell_n}{\partial p} &= 2\psi - \frac{1}{a^2} \sum_{k=1}^n \log(k) k^{2p} o_k = \sum_{k=1}^n \log(k) \Big(2 - \frac{k^{2p} o_k}{a^2} \Big) \end{aligned} \tag{6.2}$$

As expected, when p_0 is a known parameter,

$$\hat{a}^2 = \frac{1}{2n} \sum_{k=1}^n k^{2p_0} o_k, \qquad 2n \tfrac{\hat{a}^2}{a_0^2} \sim \chi^2_{2n},$$

while nothing is known about the distribution of $\hat{p}$, for small n, and for the distribution of the couple $(\hat{a}, \hat{p})$. We have the following asymptotic results, whose proof may be found in Appendix E.

Theorem 6.1. *There exists an ML estimator $(\hat{p}_n)_n$, zero of the equation (6.2), such that*

$$\hat{p}_n \xrightarrow[n\to\infty]{a.s.} p_0, \qquad \frac{\hat{p}_n - p_0}{2\sqrt{\sum_1^n \log^2(k)}} \xrightarrow[n\to\infty]{L} N(0,1).$$

Moreover,

$$2 \begin{pmatrix} \frac{\sqrt{n}}{a_0} & -\frac{\sum_{k=1}^n \log(k)}{\sqrt{n}} \\ -\frac{\sum_{k=1}^n \log(k)}{a_0 \sqrt{\sum_{k=1}^n \log^2(k)}} & \sqrt{\sum_{k=1}^n \log^2(k)} \end{pmatrix} \begin{pmatrix} \hat{a}_n - a_0 \\ \hat{p}_n - p_0 \end{pmatrix} \xrightarrow[n\to\infty]{L} \begin{pmatrix} 1 \\ -1 \end{pmatrix} Z,$$

where Z is a standard gaussian variable.

As a corollary of Theorem 6.1, the joint perfect correlation between $\hat{a}_n$ and $\hat{p}_n$ is asymptotically predicted. In Figure 2 we show this fact by simulating the processes and the maximum likelihood estimates ($\rho > 0.94$ for $n = 40$ and different values of a_0 and p_0).

Appendix A: Proofs of results of Section 2

Proof of the Theorem 2.2. By Mercer Theorem (see, e.g., [1]) we know that if $(e_n)_n$ is an orthonormal basis for the space spanned by the eigenfunctions corresponding to nonzero eigenvalues of the integral operator (2.1) then, uniformly, absolutely and in $L^2[0,1] \times [0,1]$,

$$C(s,t) = \sum_{k=0}^{\infty} e_k(t) e_k(s) \lambda_k, \tag{A.1}$$

where λ_k is the eigenvalue corresponding to e_k. By hypothesis, since $C(s,t) = \tilde{C}(|t-s|) = \tilde{C}(|t-s| \pm 1)$ by Remark 2, we get

$$\int_0^1 \tilde{C}(s) \cos(2n\pi s) ds = a_n, \qquad \int_0^1 \tilde{C}(s) \sin(2n\pi s) ds = 0, \tag{A.2}$$

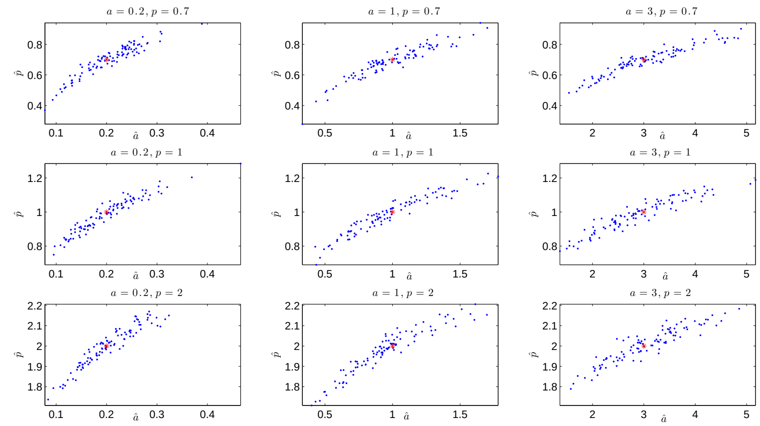

FIG 2. *Plot of* 200 $(\hat{a}, \hat{p})$*-joint simulations (blue point) of data coming from* (6.1) *for different values of* a *and* p *(red stars). In these pictures,* $n = 40$ *(see section 6.1 for notations).*

and hence

$$\tilde{C}(\tau) = a_0 + 2\sum_{n=0}^{\infty} a_n \cos(2n\pi\tau). \tag{A.3}$$

It is simple to prove that the sequence $(\mathbf{s}_n(t), \mathbf{c}_n(t))_n$ contains all the eigenfunctions of the operator (2.1). In fact,

$$\begin{aligned}
\int_0^1 C(t,\tau)\mathbf{c}_n(\tau)d\tau &= \sqrt{2}\int_0^1 \tilde{C}(s)\cos(2n\pi(t+s))dt \\
&= \mathbf{c}_n(t)\int_0^1 \cos(2n\pi s)\tilde{C}(s)ds - \mathbf{s}_n(t)\int_{-1/2}^{1/2} \sin(2n\pi s)\tilde{C}(s)ds \\
&= a_n \mathbf{c}_n(t),
\end{aligned} \tag{A.4}$$

the same relation holding when $\mathbf{c}_n(t)$ is replaced by $\mathbf{s}_n(t)$. By (A.3), we get

$$\begin{aligned}
C(s,t) = \tilde{C}(s-t) &= a_0 + \sum_{k=1}^{\infty} a_k \cos(2k\pi(s-t)) \\
&= a_0 + 2\sum_{k=1}^{\infty} a_k \cos(2k\pi s)\cos(2k\pi t) + 2\sum_{k=1}^{\infty} a_k \sin(2k\pi s)\sin(2k\pi t) \\
&= a_0 + \sum_{k=1}^{\infty} a_k \mathbf{c}_k(s)\mathbf{c}_k(t) + \sum_{k=1}^{\infty} a_k \mathbf{s}_k(s)\mathbf{s}_k(t)
\end{aligned}$$

where this equality holds uniformly, absolutely and in $L^2[0,1]\times[0,1]$ by Mercer Theorem (cfr. (A.1)).

Now, since $C(s,t)$ is a covariance function, it is positively defined, and hence $a_n \geq 0$, $\forall n$. Moreover, since $(a_n)_n \in \ell^1$, if we define $c_n = \sqrt{a_n}$, then $(c_n)_n \in \ell^2$. From Theorem 2.1 we deduce the existence of two independent sequence of independent standard gaussian variables $(Y_n)_n, (Y_n')_n$ such that in mean square, uniformly in t

$$x_t = c_0 Y_0' + \sum_{k=1}^{\infty} c_k (Y_k \mathbf{s}_k(t) + Y_k' \mathbf{c}_k(t)). \qquad \square$$

Proof of the Theorem 2.3. The sequence of gaussian processes $y_t^{(n)}$ converges to a periodical $\{y_t\}_{t\in[0,1]}$ in mean square uniformly in t, since it is a Chauchy sequence:

$$\sup_{t\in[0,1]} E[|y_t^{(n)} - y_t^{(m)}|^2] = 2\sum_{k=n}^{m} c_k^2 \underset{m,n\to\infty}{\longrightarrow} 0.$$

Hence, $E[y_t] \equiv 0$, and $Cov(y_t, y_s) = c_0^2 + \sum_{k=1}^{\infty} c_k^2 \cos(2k\pi(s-t))$ is a continuous function. Finally, $\{y_t\}_{t\in[0,1]}$ is a gaussian process, since the two sequences $(Y_n)_n$ and, $(Y_n')_n$ are formed by independent gaussian variables. $\square$

Proof of the Theorem 2.4. Necessity. Assume there exists a process $\{x_t\} \in \mathscr{H}_Z$ which generates $\{y_t\} \in \mathscr{H}_0$. The covariance function $C(s,t)$ of $\{x_t\}$ is given as in (2.2):

$$C(s,t) = c_0^2 + \sum_{k=1}^{\infty} c_k^2 \mathbf{c}_k(s)\mathbf{c}_k(t) + \sum_{k=1}^{\infty} c_k^2 \mathbf{s}_k(s)\mathbf{s}_k(t).$$

If we define $x = C(0,0) = \sum_0^{\infty} c_k^2$, $p_i = c_i^2/x$, and

$$D(s,t) = \frac{C(s,t)}{x} = p_0 + \sum_{k=1}^{\infty} p_k \mathbf{c}_k(s)\mathbf{c}_k(t) + \sum_{k=1}^{\infty} p_k \mathbf{s}_k(s)\mathbf{s}_k(t),$$

then, $x > 0$ and, by (2.3), we obtain

$$\begin{aligned} xR(s,t) &= D(s,t) - D(0,t)D(s,0) \\ &= p_0 + \sum_{k=1}^{\infty} p_k \mathbf{c}_k(s)\mathbf{c}_k(t) + \sum_{k=1}^{\infty} p_k \mathbf{s}_k(s)\mathbf{s}_k(t) + \\ &\qquad - \Big(p_0 + \sum_{k=1}^{\infty} p_k \mathbf{c}_k(s)\Big)\Big(p_0 + \sum_{k=1}^{\infty} p_k \mathbf{c}_k(t)\Big). \end{aligned} \tag{A.5}$$

(A.5) and (2.4) give

$$xr_{kj}^{ss} = \begin{cases} p_k & \text{if } k = j > 0 \\ 0 & \text{if } k \neq j \end{cases} \tag{A.6}$$

$$xr_{kj}^{cc} = \begin{cases} p_k - p_k^2 & \text{if } k = j \geq 0 \\ -p_k p_j & \text{if } k \neq j \end{cases} \tag{A.7}$$

$$r_{kj}^{sc} = r_{kj}^{cs} = 0$$

Since $\sum_0^\infty p_k = 1$, if $\bar{r} = \sum_{k=1}^\infty r_{kk}^{ss}$, we obtain by (A.6)

$$x\bar{r} = 1 - p_0.$$

Assume $\bar{r} = 0$, then $p_0 = 1$, which is absurd since $R(s,t) \neq 0$. Hence $\bar{r} > 0$, and we define

$$x = \frac{\bar{r} - r_{00}^{cc}}{\bar{r}^2}, \tag{A.8}$$

Thesis follows by combining (A.8) and (A.7).

Sufficiency. Given the matrices of the $2-D$ Fourier series as in the theorem assumption, set $x > 0$ as in (A.8). Define

$$p_k = r_{kk}^{ss} \frac{\bar{r} - r_{00}^{cc}}{\bar{r}^2}, \qquad p_0 = \frac{r_{00}^{cc}}{\bar{r}}.$$

Then $(p_k)_{k\geq 0}$ is a non-negative sequence such that $\sum_k p_k = 1$. Define

$$x_t = \sqrt{xp_0}Y_0' + \sum_{k=1}^n \sqrt{xp_k}(Y_k \mathbf{s}_k(t) + Y_k' \mathbf{c}_k(t)).$$

By Theorem 2.2, we have

$$C(s,t) = x\Big(p_0 + \sum_{k=1}^\infty p_k \mathbf{c}_k(s)\mathbf{c}_k(t) + \sum_{k=1}^\infty p_k \mathbf{s}_k(s)\mathbf{s}_k(t)\Big).$$

It is straightforward to check that (A.6) and (A.7) hold. The fact that the solution is unique follows immediately from the necessary condition. □

Appendix B: Proofs of results of Section 3

Proof of the Theorem 3.1. We recall that $E(y_t) = 0$, and $\forall 0 \leq s,t \leq 1, Cov(y_t, y_s) = \min(s,t)(1 - \max(s,t))$. A straightforward calculation yields, for any $k \geq 1$,

$$\begin{aligned} r_{kk}^{cc} &= \int_0^1 \int_0^1 Cov(y_t, y_s)\mathbf{c}_k(s)\mathbf{c}_k(t) ds\, dt \\ &= \frac{1}{(2k\pi)^2} = \int_0^1 \int_0^1 Cov(y_t, y_s)\mathbf{s}_k(s)\mathbf{s}_k(t) ds\, dt = r_{kk}^{ss}, \end{aligned}$$

which contradicts the necessary condition for the existence of a process $\{x_t\} \in \mathscr{H}$ which generates $\{y_t\}_{t\in[0,1]}$ given in the Theorem 2.4. □

Proof of the Theorem 3.2. Assume there exists a process $\{\tilde{x}_t, t \in [0,2]\} \in \mathscr{H}(0,2)$ which generates an extension of $\{y_t\}$ on $[0,2]$. Recall that the extension is on $[0,2]$ and thus, by (2.2), the covariance function of $C(s,t)$ is given by:

$$C(s,t) = c_0^2 + \sum_{k=1}^\infty c_k^2 \cos(k\pi s)\cos(k\pi t) + \sum_{k=1}^\infty c_k^2 \sin(k\pi s)\sin(k\pi t).$$

The covariogram function is $\tilde{C}(\delta) = p_0 + \sum_{k=1}^{\infty} p_k \cos(k\pi\delta)$, where $p_l = c_l^2 \geq 0$. Since $y_1 = 0$, by (2.3), we obtain

$$
\begin{aligned}
0 \equiv R(s,1) &= \tilde{C}(1-s) - \frac{\tilde{C}(s)\tilde{C}(1)}{\tilde{C}(0)} \\
&= \Big(p_0 + \sum_{1}^{\infty} (-)^k p_k \cos(k\pi s)\Big) - \frac{\Big(p_0 + \sum_{k=1}^{\infty} p_k \cos(k\pi s)\Big)\Big(\sum_0^{\infty} (-)^k p_k\Big)}{\sum_0^{\infty} p_k}.
\end{aligned}
$$

The integration on $[0,2]$ with respect to $\cos(2m\pi s)$, gives

$$
p_{2m}\Big(\sum_{k=0}^{\infty} p_{2k+1}\Big) = 0, \qquad \forall m \geq 0,
$$

and hence either all the odd terms $(p_{2k+1})_k$ or all the even terms $(p_{2m})_m$ are null, since the p_k's are nonnegative.

Therefore, the possible extensions of $\{y_t, t \in [0,1]\}$ are either 1–antiperiodic or 1–periodic (see Remark 3), which implies that the possible extensions of the covariance function of $\{y_t, t \in [0,1]\}$ to $[0,2]^2$ is given by

$$
\tilde{R}(s,t) = \begin{cases} R(s,t) & \text{if } 0 \leq s,t \leq 1; \\ (-)^i R(s,t-1) & \text{if } 0 \leq s \leq 1 \text{ and } 1 < t \leq 2; \\ (-)^i R(s-1,t) & \text{if } 0 \leq t \leq 1 \text{ and } 1 < s \leq 2; \\ R(s-1,t-1) & \text{if } 1 < s,t \leq 2. \end{cases}
$$

where $i = 1$ relates to the antiperiodic extension and $i = 2$ to the periodic one. Theorem 2.4 states that if each of this two extensions may be generated by a (unique) process $\{\tilde{x}_t^{(i)}, t \in [0,2]\} \in \mathscr{H}_Z(0,2)$, $i = 1,2$.

Now, we prove that it is not possible to have both the extensions at the same time. In fact, this fact would imply that both the functions

$$
\tilde{C}^{(1)}(\delta) = \sum_{k=1}^{\infty} p_{2k+1} \cos((2k+1)\pi\delta), \qquad \tilde{C}^{(2)}(\delta) = \sum_{k=0}^{\infty} p_{2k} \cos(2k\pi\delta)
$$

generates $R(s,t) = \tilde{C}^{(i)}(t-s) - \frac{\tilde{C}^{(i)}(t)\tilde{C}^{(i)}(s)}{\tilde{C}^{(i)}(0)}$ by (2.3).

Since $\tilde{C}^{(1)}(\frac{1}{2}) = 0$, we get

- if $s \in [0, 1/2]$,

$$
\begin{aligned}
\Big(\sum_{k=0}^{\infty} p_{2k}\cos(2k\pi s)\Big) - \frac{\Big(\sum_0^\infty (-)^k p_{2k}\Big)\Big(\sum_{k=0}^\infty p_{2k}\cos(2k\pi s)\Big)}{\sum_0^\infty p_{2k}} \\
&= \tilde{C}^{(2)}(s) - \frac{\tilde{C}^{(2)}(\frac{1}{2})\tilde{C}^{(2)}(\frac{1}{2}+s)}{\tilde{C}^{(2)}(0)} \\
&= R\Big(\frac{1}{2}, \frac{1}{2}+s\Big) \\
&= \tilde{C}^{(1)}(s) - \frac{\tilde{C}^{(1)}(\frac{1}{2})\tilde{C}^{(1)}(\frac{1}{2}+s)}{\tilde{C}^{(1)}(0)} \\
&= \tilde{C}^{(1)}(s) = \sum_{k=1}^{\infty} p_{2k+1}\cos((2k+1)\pi s)
\end{aligned} \tag{B.1}
$$

- Again, since $2k\pi(\frac{1}{2}+s) = -2k\pi(\frac{3}{2}-s) \mod 2\pi$, when $s \in (1/2, 1]$, we obtain

$$
\begin{aligned}
\Big(\sum_{k=0}^{\infty} p_{2k}\cos(2k\pi s)\Big) - \frac{\Big(\sum_0^\infty (-)^k p_{2k}\Big)\Big(\sum_{k=0}^\infty p_{2k}\cos(2k\pi s)\Big)}{\sum_0^\infty p_{2k}} \\
&= \tilde{C}^{(2)}(s) - \frac{\tilde{C}^{(2)}(\frac{1}{2})\tilde{C}^{(2)}(\frac{1}{2}+s)}{\tilde{C}^{(2)}(0)} \\
&= \tilde{C}^{(2)}(1-s) - \frac{\tilde{C}^{(2)}(\frac{1}{2})\tilde{C}^{(2)}(\frac{3}{2}-s)}{\tilde{C}^{(2)}(0)} \\
&= R\Big(\frac{1}{2}, \frac{3}{2}-s\Big) \\
&= \tilde{C}^{(1)}(1-s) - \frac{\tilde{C}^{(1)}(\frac{1}{2})\tilde{C}^{(1)}(\frac{3}{2}-s)}{\tilde{C}^{(1)}(0)} \\
&= \tilde{C}^{(1)}(s) = \sum_{k=1}^{\infty} p_{2k+1}\cos((2k+1)\pi s)
\end{aligned} \tag{B.2}
$$

By integrating (B.1) and (B.2) on $[0, 1]$ with respect to $\cos(2(k+1)\pi s)$, we obtain $p_{2k+1} = 0$ for any k, which implies $C^{(1)} \equiv 0$. □

Proof of the Theorem 3.3. First, we prove that $\{\tilde{x}_t, t \in [0, 2]\}$ given in (3.1) generates the antiperiodic extension of the brownian bridge to $[0, 2]$. In fact, it is sufficient to note that

- the function

$$
C^{(1)}(\delta) = \begin{cases} R(\frac{1}{2}, \frac{1}{2}+\delta) = \frac{1}{4} - \frac{\delta}{2} & \text{if } 0 \le \delta \le \frac{1}{2}; \\ R(\frac{1}{2}, \frac{3}{2}-\delta) = \frac{1}{4} - \frac{\delta}{2} & \text{if } \frac{1}{2} < \delta \le 1; \\ -C^{(1)}(\delta-1) = \frac{1}{4} - \frac{2-\delta}{2} & \text{if } 1 < \delta \le 2. \end{cases}
$$

is a positive defined function on $[0, 2]$, since the well-known Fourier series expansion of $f(x) = |x|$ on $[-1, 1]$ gives

$$C^{(1)}(\delta) = 4\sum_{k=1}^{\infty} \frac{\cos(\pi(2k+1)\delta)}{(\pi(2k+1))^2};$$

- the function $C^{(1)}$ may be extended to a continuous 2-periodic function;
- the process $\{\tilde{x}_t, t \in [0, 2]\}$ has covariogram function $C^{(1)}$ by Theorem 2.2 and Theorem 2.3 (extended on $[0, 2]$);
- by (2.3), $R(s, t) = s(1-t)$ for $0 \le s \le t \le 1$, i.e. $\{\tilde{x}_t, t \in [0, 2]\}$ generates the brownian bridge on $[0, 1]$.

Then, since (3.1) is 1–antiperiodic, the process $\{\tilde{y}_t^{(1)}, t \in [0, 2]\} \in \mathscr{H}_0(0, 2)$ generated by $\{\tilde{x}_t, t \in [0, 2]\}$ is

$$\tilde{y}_t = \begin{cases} y_t & \text{if } 0 \le t \le 1; \\ -y_{t-1} & \text{if } 1 < t \le 2. \end{cases}$$

To prove the uniqueness, by Lemma 3.2, we must prove that the periodic extension of the brownian bridge does not belong to $\mathscr{H}_0(0, 2)$. Assume the converse, there would be a 1–periodic periodic process which generates the brownian bridge, which contradicts Theorem 3.1. □

Appendix C: Proof of the Theorem 4.1

The case $x_t \equiv k$ is obvious. Let $C(t, s) = \tilde{C}(t-s)$ be the covariogram function of $\{x_t\}_{t\in[0,1]}$ (see (2.2) for its expansion). Since $x_t \equiv k \iff \tilde{C}(0) = 0$, we assume, without loss of generalities, that $\tilde{C}(0) = 1$.

A straightforward computation gives that, if $\{y_t\}_{t\in[0,1]} \in \mathscr{H}_0$ is generated by $\{x_t\}_{t\in[0,1]} \in \mathscr{H}$, then $\{y_t\}_{t\in[0,1]}$ is a gaussian process with null expectation and continuous covariance function

$$R(t, s) = \tilde{C}(t-s) - \frac{\tilde{C}(t)\tilde{C}(s)}{\tilde{C}(0)} = \tilde{C}(t-s) - \tilde{C}(t)\tilde{C}(s). \tag{C.1}$$

Hence, given the covariogram function $C(s, t) = \tilde{C}(t-s)$ of the generating process $\{x_t\}_{t\in[0,1]}$, we need to study the spectrum of the operator (2.1), where C is replaced by R given in (C.1).

As in (A.3) and (2.2), we write $\tilde{C}(t) = a_0 + 2\sum_{n=1}^{\infty} a_n \cos(2n\pi t)$ with $1 = a_0 + 2\sum_{n=1}^{\infty} a_n$ since $\tilde{C}(0) = 1$. Let $f(t)$ an eigenfunction of (C.1); for expansion theorem (see [1]) we have in $L^2[0, 1]$,

$$f(t) = f_0 + \sum_{n=1}^{\infty} f_n^c \mathbf{c}_n(t) + f_n^s \mathbf{s}_n(t). \tag{C.2}$$

where $f_0 = \int_0^1 f(\tau)d\tau$, $f_n^c = \int_0^1 \mathbf{c}_n(\tau)f(\tau)d\tau$ and $f_n^s = \int_0^1 \mathbf{s}_n(\tau)f(\tau)d\tau$. Let's look for the eigenvalue related to f:

$$\int_0^1 R(s,t)f(t)dt = \int_0^1 \tilde{C}(t-s)f(t)dt - \tilde{C}(s)\int_0^1 \tilde{C}(t)f(t)dt = \tilde{a}f(s). \quad \text{(C.3)}$$

Substituting (C.2) into (C.3), and integrating with the results in (A.2) and (A.4), yields

$$a_0 f_0 + \sum_{n=1}^{\infty} a_n(f_n^c \mathbf{c}_n(s) + f_n^s \mathbf{s}_n(s)) - \tilde{C}(s)\Big(a_0 f_0 + \sqrt{2}\sum_{n=1}^{\infty} a_n f_n^c\Big) = \tilde{a}f(s). \quad \text{(C.4)}$$

C.1. $\mathbf{s}_n(s)$ *eigenfunctions*

For any $a_n \neq 0$, it is straightforward to see that $f(s) = \mathbf{s}_n(s)$ is an eigenfunction, by a direct substitution in (C.4), and that $\tilde{a} = a_n$. Moreover, we are going to state more: the only eigenfunctions which contains some $f_k^s \neq 0$ are indeed $\mathbf{s}_n(s)$ (when $a_n \neq 0$).

Assume that $\exists k$: $f_k^s \neq 0$ and, by contradiction, $f(t) \neq \mathbf{s}_k(t)$.
By multiplying both members of (C.4) by $\mathbf{s}_k(s)$ and integrating, we obtain $a_k f_k^s = \tilde{a} f_k^s$, i.e., $a_k = \tilde{a}$. Since $a_k \neq 0$, then $\mathbf{s}_k(t)$ is an eigenfunction. This eigenfunction is orthogonal to $f(s)$ by Mercer Theorem, and hence

$$0 = \int_0^1 \mathbf{s}_k(s)f(s)ds = f_k^s.$$

Summing up, for any $a_n \neq 0$, $\mathbf{s}_n(t)$ is an eigenfunction associated to $\tilde{a} = a_n$, and the other eigenfunctions do not contain the terms in $(\mathbf{s}_n(t))_n$ (they are even function).

C.2. The other eigenfunctions of (C.4)*.*

To conclude the proof, we should find another sequence of eigenfunctions with eigenvalues $(\tilde{a}_n)_n \asymp (a_n)_n$. We will first obtain a simple result on the coefficients of the eigenfunctions. Then we will introduce the multiplicity of the eigenvectors $(a_n)_n$ in order to conclude the proof accordingly.

The other eigenfunction takes the form $f(t) = f_0 + \sum_{k=1}^{\infty} f_k^c \mathbf{c}_k(t)$. By multiplying both members of (C.4) by $\mathbf{c}_n(s)$ and integrating, we obtain

$$\begin{cases} a_0 f_0 - a_0(a_0 f_0 + \sqrt{2}\sum_{k=1}^{\infty} a_k f_k^c) = \tilde{a} f_0, & n = 0; \\ a_n f_n^c - \sqrt{2}a_n(a_0 f_0 + \sqrt{2}\sum_{k=1}^{\infty} a_k f_k^c) = \tilde{a} f_n^c, & n > 0. \end{cases} \quad \text{(C.5)}$$

As an immediate consequence, $(a_n = 0) \Rightarrow (f_n^c = 0)$.

Lemma C.1. *$(f_n^c)_{n\in\mathbb{N}} \in \ell^1$, and $f_0 + \sqrt{2}\sum_{n=1}^{\infty} f_n^c = 0$.*

Proof. Recall that $a_n \geq 0$, and that $a_0 + 2\sum_{n=1}^\infty a_n = \tilde{C}(0) = 1$. For $n > 0$, by (C.5), we have

$$|f_n^c| \leq \frac{a_n|f_n^c| - \sqrt{2}a_n(a_0|f_0| + \sqrt{2}\sum_{k=1}^\infty a_k|f_k^c|)}{\tilde{a}},$$

and since $(a_k|f_k^c|)_k \in \ell^1$ (as a product of two ℓ^2 sequences), and $(a_n)_n \in \ell^1$, we obtain the first part of the thesis. By (C.5) and $a_0 + 2\sum_{n=1}^\infty a_n = \tilde{C}(0) = 1$, we get

$$\begin{aligned} f_0 + \sqrt{2}\sum_{n=1}^\infty f_n^c &= \frac{a_0 f_0 - a_0(a_0 f_0 + \sqrt{2}\sum_{k=1}^\infty a_k f_k^c)}{\tilde{a}} \\ &\quad + \sqrt{2}\sum_{n=1}^\infty \frac{a_n f_n^c - \sqrt{2}a_n(a_0 f_0 + \sqrt{2}\sum_{k=1}^\infty a_k f_k^c)}{\tilde{a}} \\ &= \frac{a_0 f_0 + \sqrt{2}\sum_{n=1}^\infty a_n f_n^c}{\tilde{a}} \\ &\quad - \frac{a_0 f_0 + \sqrt{2}\sum_{k=1}^\infty a_k f_k^c}{\tilde{a}}\Big(a_0 + 2\sum_{n=1}^\infty a_n\Big) = 0. \square \end{aligned}$$

Definition C.1 (Multiplicity and support)**.** Given $(a_n)_n$, we define the *support* $S_{\tilde{a}}$ of $\tilde{a}$:

$$S_{\tilde{a}} = \{k \colon a_k = \tilde{a}\}.$$

The *multiplicity* $m_{\tilde{a}}$ of a number $\tilde{a} > 0$ is the cardinality of $S_{\tilde{a}}$:

$$m_{\tilde{a}} = \#\{k \colon a_k = \tilde{a}\}.$$

It is clear that $m_{\tilde{a}} < \infty$ because $(a_n)_n \in \ell^1$.

Lemma C.2. *If $m_{\tilde{a}} = k > 0$, then there are exactly $k-1$ orthogonal eigenfunctions of R related to $\tilde{a}$. Moreover for anyone of these $k-1$ eigenfunctions,*

$$n \notin S_{\tilde{a}} \qquad \Longrightarrow \qquad f_n^c = 0.$$

Proof. Let $\tilde{a} > 0$ be such that $m_{\tilde{a}} > 1$.

It is simple to prove that there always exist $m_{\tilde{a}} - 1$ orthogonal eigenfunctions related to $\tilde{a}$ with $f_n^c = 0$ if $a_n \notin S_{\tilde{a}}$. We have two possibilities:

- $0 \in S_{\tilde{a}}$ or, equivalently, $a_0 = \tilde{a}$. In this case, (C.5) is equivalent to the following system

$$\begin{cases} f_n^c = 0, & n \notin S_{\tilde{a}} \\ \tilde{a}\big(f_0 + \sqrt{2}\sum_{n \in S_{\tilde{a}} \setminus \{0\}} f_n^c\big) = 0. \end{cases}$$

- $0 \notin S_{\tilde{a}}$. In this case, (C.5) is equivalent to the following system

$$\begin{cases} f_n^c = 0, & n \notin S_{\tilde{a}} \\ \tilde{a}\big(\sum_{n \in S_{\tilde{a}}} f_n^c\big) = 0. \end{cases}$$

In both cases, there exist a $k-1$-dimensional orthogonal basis for the solution system.

We now need to prove that there are not other eigenfunctions related to $\tilde{a}$. Assume that $f^c_{\bar{n}} \neq 0$. We recall that this fact implies $a_{\bar{n}} \neq 0$. If $\bar{n} = 0$, from (C.5) we have that

$$\begin{cases} \frac{(a_0-\tilde{a})f_0}{a_0} = a_0 f_0 + \sqrt{2}\sum_{k=1}^\infty a_k f^c_k, \\ \frac{(a_n-\tilde{a})f^c_n}{\sqrt{2}a_n} = a_0 f_0 + \sqrt{2}\sum_{k=1}^\infty a_k f^c_k, \quad n \in S_{\tilde{a}} \end{cases}$$

The second equation shows that $a_0 f_0 + \sqrt{2}\sum_{k=1}^\infty a_k f^c_k = 0$, since $a_n = \tilde{a}$, and hence $a_0 = \tilde{a}$, which means that $\bar{n} \in S_{\tilde{a}}$. Analogously, if $\bar{n} \neq 0$, from (C.5) we can prove that $\bar{n} \in S_{\tilde{a}}$, that completes the proof. □

Let $(a_{(n)})_n$ be the decreasing reordering of the sequence $(a_n)_n$, positive and without repetition: $a_{(1)} > a_{(2)} > \cdots > a_{(n)} > \cdots$ and $\forall a_n > 0$, exists k such that $a_n = a_{(k)}$. To conclude the proof, we must find a sequence of eigenvalues $(\tilde{a}_n)_n$ such that $(\tilde{a}_n)_n \asymp (a_{(n)})_n$, and $\tilde{a}_n \neq a_{(k)}$, for any n and k.

Lemma C.3. *For each $n \in \mathbb{N}$, there exists a unique eigenvalue $\tilde{a}_n$ such that $a_{(n)} > \tilde{a}_n > a_{(n+1)}$. Moreover, $m_{\tilde{a}_n} = 1$.*

Proof. We have already observed that $a_0 f_0 + \sqrt{2}\sum_{k=1}^\infty a_k f^c_k = 0$ implies, for any n, $a_n = \tilde{a}$ or $f^c_n = 0$. Hence, without loss of generalities, we assume $a_0 f_0 + \sqrt{2}\sum_{k=1}^\infty a_k f^c_k = c \neq 0$ and we continue the proof. From (C.5), we obtain

$$f_0 = c\frac{a_0}{a_0 - \tilde{a}}, \qquad f^c_n = c\frac{\sqrt{2}a_n}{a_n - \tilde{a}}. \tag{C.6}$$

These relations with, again, $a_0 f_0 + \sqrt{2}\sum_{n=1}^\infty a_n f^c_n = c$, imply

$$\frac{a_0^2}{a_0 - \tilde{a}} + 2\sum_{n=1}^\infty \frac{a_n^2}{a_n - \tilde{a}} = 1. \tag{C.7}$$

We are going to show that there exists a unique solution $\tilde{a}_n$ of (C.7) such that $a_{(n)} > \tilde{a}_n > a_{(n+1)}$. These are the wanted eigenvalues, whose corresponding eigenfunctions' expansion is given in (C.6).

Let us consider the series

$$S(x) = \frac{a_0^2}{a_0 - x} + 2\sum_{n=1}^\infty \frac{a_n^2}{a_n - x}$$

and the derivative series

$$S'(x) = \frac{a_0^2}{(a_0 - x)^2} + 2\sum_{n=1}^\infty \frac{a_n^2}{(a_n - x)^2}$$

then they converge absolutely in each compact not containing $(a_n)_n$. We have that

$$\mathrm{dom}(S) = \mathrm{dom}(s) = \cup_n (a_{(n+1)}, a_{(n)}), \qquad S'(x) = s(x), \forall x \in \mathrm{dom}(S).$$

Moreover for each n,

$$\lim_{x\to a^+_{(n+1)}} S(x) = -\infty, \qquad \lim_{x\to a^-_{(n)}} S(x) = +\infty, \qquad S'(x) > 0, \forall x \in (a_{(n+1)}, a_{(n)}).$$

Hence, there exists a unique $\tilde{a}_n \in (a_{(n+1)}, a_{(n)})$ such that $S(\tilde{a}_n) = 1$, i.e. for which (C.7) holds. The unique corresponding eigenfunction is given by (C.6), that implies also $m_{\tilde{a}_n} = 1$:

$$f(t) = \frac{a_0}{a_0 - \tilde{a}_n} + \sqrt{2}\sum_{n=1}^{\infty} \frac{a_n}{a_n - \tilde{a}_n}\mathbf{c}_n(t).$$

To complete the proof, we show that there are not eigenvalues greater then $a_{(1)} = \max_n a_n$ or smaller than any $a_n > 0$.

In fact, if we assume that there exists an eigenvalue $\hat{a} > \max a_n$, then (C.6) shows that the sequence $(f^c_k)_k$ is made of either nonnegative or nonpositive numbers, that together with Lemma C.1 implies $f^c_k = 0$, for any f.

In the same way it can be shown that there are no eigenvalues smaller than any $a_n > 0$. □

Appendix D: Proofs of results of Section 5

We simply deduce the results basing on the fact that if $Y \approx N(0, \sigma^2)$, then $E(|Y|^p) = \sigma^p \frac{2^{\frac{p}{2}}\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}$, (see, e.g., [11]).

Proof of the Theorem 5.2. . Observe that

$$E(|x_{t+h} - x_t|^2) = E(x_t^2 + x_{t+h}^2 - 2x_{t+h}x_t) =$$

$$= R(t+h, t+h) + R(t,t) - 2R(t+h,t)$$

but there exists an M such that

$$|R(s+\delta_1, t+\delta_2) - R(s,t)| \leq M\|(\delta_1, \delta_2)\|^\alpha$$

and so there exists a D such that

$$E(|x_{t+h} - x_t|^2) \leq D|h|^\alpha.$$

Then the thesis follows. □

Proof of the Theorem 5.3. The first part of the theorem is a simple calculation. The second holds because, using theorem 5.1 we have

$$E(|x_{t+h} - x_t|^2) = E(E(|x_{t+h} - x_t|^2|x_0)) =$$

$$= E(E((x_{t+h} - x_0) - (x_t - x_0))^2|x_0)) =$$

$$= R(t+h, t+h) + R(t,t) - 2R(t+h,t) \leq D|h|^\alpha.$$

□

Proof of the Theorem 5.5. It is clear that

$$\partial^2 \tilde{C}(\delta) = 2\partial^2 \sum_{k=1}^{\infty} c_k^2 \cos(2k\pi(\delta)) = -2\sum_{k=1}^{\infty} (2\pi)^2 k^2 c_k^2 \cos(2k\pi(\delta))$$

and that $\partial^2 \tilde{C} \in C^{0,\alpha}([0,1])$, for some $0 < \alpha \leq 1$. Moreover we have that uniformly in t and in mean square

$$x_t = c_0 Y_0' + \sum_{k=1}^{\infty} c_k (Y_k \mathbf{s}_k(t) + Y_k' \mathbf{c}_k(t)).$$

and, from Theorem 2.2, there also exist a stochastic process in $\mathscr{H}$ such that uniformly in t and in mean square

$$\tilde{x}_t = 2\pi \sum_{k=1}^{\infty} k c_k (Y_k \mathbf{c}_k(t) - Y_k' \mathbf{s}_k(t)),$$

which has covariogram function belonging to $C^{0,\alpha}([0,1])$ given by

$$\tilde{\tilde{C}}(\delta) = 2\sum_{k=1}^{\infty} (2\pi)^2 k^2 c_k^2 \cos(2k\pi(\delta)).$$

If we define

$$y_t^{(n)} := c_0 Y_0' + \sum_{k=1}^{n} c_k (Y_k \mathbf{s}_k(t) + Y_k' \mathbf{c}_k(t))$$

$$\tilde{y}^{(n)}(t) := 2\pi \sum_{k=1}^{n} k c_k (Y_k \mathbf{c}_k(t) - Y_k' \mathbf{s}_k(t)),$$

than $y_t^{(n)} = y_0^{(n)} + \int_0^t \tilde{y}_\tau^{(n)} d\tau$, a.s. for any n, while for each fixed t, in mean square we have $\int_0^t \tilde{y}_\tau^{(n)} d\tau \to \int_0^t \tilde{x}_\tau d\tau$. Since

$$\sqrt{E\Big((x_t - x_0 - \int_0^t \tilde{x}_\tau d\tau)^2\Big)} \leq \sqrt{E\Big((x_t - y_t^{(n)})^2\Big)} + \sqrt{E\Big((y_0^{(n)} + \int_0^t \tilde{y}_\tau^{(n)} d\tau - x_0 - \int_0^t \tilde{x}_\tau d\tau)^2\Big)} \underset{n\to\infty}{\longrightarrow} 0,$$

it follows that a.s. $x_t = x_0 + \int_0^t \tilde{x}_\tau d\tau$. By Theorem 5.2 we know that almost all trajectory path of $\tilde{x}_t$ belongs to $C^{0,\beta}([0,1])$, with $\beta < \frac{\alpha}{2}$, and thesis follows. □

Appendix E: Proofs of results of Section 6

Proof of Theorem 6.1. Assume p_0 be the true parameter, we may define a sequence of i.i.d. random variables $(Z_k)_k$ in the following way:

$$Z_k \sim \frac{k^{2p_0} o_k}{a_0^2} \sim \chi_2^2 \sim \exp(\tfrac{1}{2}). \tag{E.1}$$

Equation (6.2), as a function of (p, p_0) and $(Z_k)_k$, becomes

$$\frac{\partial \ell_n}{\partial p} = \sum_{k=1}^n \log(k)\Big(2 - k^{2(p-p_0)} Z_k\Big).$$

With the notation of [5, pagg. 155-161], we have

$$\begin{aligned} I_n(p) &= \sum_1^n \log^2(k) E\Big((2 - k^{2(p-p_0)} Z_k)^2 | Z_1, \ldots, Z_{k-1}\Big) \\ &= \sum_1^n \log^2(k) 2(1 + (1 - 2k^{2(p-p_0)})^2), \\ J_n(p) &= -\frac{2}{a_0^2} \sum_{k=1}^n \log^2(k) k^{2p} o_k = -2 \sum_{k=1}^n \log^2(k) k^{2(p-p_0)} Z_k \end{aligned}$$

and, in particular,

$$I_n(p_0) = 4 \sum_1^n \log^2(k), \tag{E.2}$$

$$J_n(p_0) = -2 \sum_1^n \log^2(k) Z_k. \tag{E.3}$$

The thesis is a consequence of [5, pagg. 155-161], where the Assumption 1 and Assumption 2 on page 160 guarantee the existence of a ML estimator $(\hat{p}_n)_n$ such that

$$\hat{p}_n \xrightarrow[n\to\infty]{a.s.} p_0, \qquad \frac{\hat{p}_n - p_0}{\sqrt{I_n(p_0)}} \xrightarrow[n\to\infty]{L} N(0,1).$$

Check of [5, Assumption 1, pagg. 160]. The fact that $I_n(p_0) \xrightarrow[n\to\infty]{a.s.} \infty$ is a consequence of (E.2). As $I_n(p_0) = E(I_n(p_0))$, then $I_n(p_0)/E(I_n(p_0)) \to 1$ uniformly on compacts. By (E.2) and (E.3), we have

$$\frac{J_n(p_0)}{I_n(p_0)} = \frac{-2\sum_1^n \log^2(k) Z_k}{4\sum_1^n \log^2(k)},$$

and hence, by (E.1), we have

$$E\Big(\frac{J_n(p_0)}{I_n(p_0)}\Big) = 1, \qquad Var\Big(\frac{J_n(p_0)}{I_n(p_0)}\Big) = \frac{\sum_1^n \log^4(k)}{(\sum_1^n \log^2(k))^2}$$

Since, for $n \geq 4$,

$$\frac{\log^2(n)}{\sum_1^n \log^2(m)} \leq \frac{1}{\sum_{n/2}^n \big(\frac{\log(n/2)}{\log(n)}\big)^2} \leq \frac{1}{\sum_{n/2}^n \big(\frac{1}{2}\big)^2} \leq \frac{8}{n} \tag{E.4}$$

then $\sum_{n=1}^{\infty}\big(\frac{\log^2(n)}{\sum_1^n \log^2(m)}\big)^2 < \infty$, and hence $Var\big(\frac{J_n(p_0)}{I_n(p_0)}\big) \to 0$ by Kronecker's Lemma, which ensures that $I_n(p_0)/E(I_n(p_0)) \to -1$ in probability uniformly on compacts.

Check of [5, Assumption 2, pagg. 160]. Since, for any p, $E_p(I_n(p))$ does not change, then Assumption 2.i) is automatically satisfied.

Now, if $|p_n - p_0| \leq \delta/\sqrt{I_n(p_0)}$, we get

$$|J_n(p_n) - J_n(p_0)| \leq 2\sum_{k=1}^{n} \log^2(k)\big(k^{\frac{\delta}{\sqrt{\sum_1^n \log^2(m)}}} - 1\big) Z_k \tag{E.5}$$

$$|I_n(p_n) - I_n(p_0)| \leq \sum_{k=1}^{n} \log^2(k) 8 k^{2\frac{\delta}{\sqrt{I_n(p_0)}}} (k^{2\frac{\delta}{\sqrt{I_n(p_0)}}} - 1)$$

Note that, since $k \leq n$, we have

$$1 \leq k^{2\frac{\delta}{\sqrt{I_n(p_0)}}} \leq e^{2\frac{\delta}{\sqrt{I_n(p_0)}}\log(n)} \leq \exp(2\delta)$$

and hence, for sufficient large n and $k \leq n$, since

$$k^{2\frac{\delta}{\sqrt{I_n(p_0)}}} - 1 \leq C_0 2 \frac{\delta}{\sqrt{I_n(p_0)}} \log(k), \tag{E.6}$$

we obtain

$$\Big|\frac{I_n(p_n) - I_n(p_0)}{I_n(p_0)}\Big| \leq \frac{C_1 \frac{\sum_{k=1}^n \log^3(k)}{\sqrt{\sum_1^n \log^2(k)}}}{4\sum_1^n \log^2(k)} = C_2 \sum_{k=1}^{n} \Big(\frac{\log^2(k)}{\sum_1^n \log^2(m)}\Big)^{\frac{3}{2}}$$

By (E.4), then $\sum_{n=1}^{\infty}\big(\frac{\log^2(n)}{\sum_1^n \log^2(m)}\big)^{\frac{3}{2}} < \infty$, and hence, by Kronecker's Lemma, we get Assumption 2.ii), namely

$$\Big|\frac{I_n(p_n) - I_n(p_0)}{I_n(p_0)}\Big| \to 0.$$

The last Assumption 2.iii) requires that

$$\frac{J_n(p_n) - J_n(p_0)}{I_n(p_0)} \to 0, \qquad \text{a.s.}$$

To check this, we first note that

$$\frac{\sum_{k=1}^n \log^2(k)\big(k^{\frac{\delta}{\sqrt{\sum_1^n \log^2(m)}}} - 1\big)}{\sum_1^n \log^2(k)} \to 0,$$

as a consequence of Kronecker's Lemma, (E.6) and (E.4). Then

$$\Big|E\Big(\frac{J_n(p_n) - J_n(p_0)}{I_n(p_0)}\Big)\Big| \leq \frac{E(|J_n(p_n) - J_n(p_0)|)}{I_n(p_0)} \to 0,$$

and hence, a sufficient condition for $\frac{J_n(p_n)-J_n(p_0)}{I_n(p_0)} \to 0$ to hold, is that

$$Var\Big(\frac{J_n(p_n)-J_n(p_0)}{I_n(p_0)}\Big) \to 0. \tag{E.7}$$

By (E.5), since $Var(X_k)=4$, we obtain

$$Var(J_n(p_n)-J_n(p_0)) \le 8\sum_{k=1}^n \log^4(k)\big(k^{\frac{\delta}{\sqrt{\sum_1^n \log^2(m)}}}-1\big)^2.$$

Again, by (E.6), we obtain

$$Var\Big(\frac{J_n(p_n)-J_n(p_0)}{I_n(p_0)}\Big) \le \frac{C_1\frac{\sum_{k=1}^n \log^6(k)}{\sum_1^n \log^2(k)}}{\Big(4\sum_1^n \log^2(k)\Big)^2} = C_2\sum_{k=1}^n \Big(\frac{\log^2(k)}{\sum_1^n \log^2(m)}\Big)^3$$

As above, by (E.4) and Kronecker's Lemma, we obtain (E.7).

We sketch the second part of the proof, with the notation of [6, pag.191]. If we define

$$\mathbf{G}_n(\boldsymbol{\theta}) = \begin{cases} G_n^{(1)}(a,p) = \frac{1}{a}\sum_{k=1}^n \Big(\frac{k^{2p}}{a^2}o_k - 2\Big) = \frac{1}{a}\sum_{k=1}^n \Big(\frac{k^{2(p-p_0)}a_0^2}{a^2}Z_k - 2\Big) \\ G_n^{(2)}(a,p) = \sum_{k=1}^n \log(k)\Big(2-\frac{k^{2p}}{a^2}o_k\Big) = \sum_{k=1}^n \log(k)\Big(2-\frac{k^{2(p-p_0)}a_0^2}{a^2}Z_k\Big) \end{cases}$$

and

$$\mathbf{H}_n^{-1}(a_0,p_0) = \begin{pmatrix} \frac{a_0}{2\sqrt{n}} & 0 \\ 0 & \frac{1}{2\sqrt{\sum_{k=1}^n \log^2(k)}} \end{pmatrix}$$

it is simple to state that

$$\mathbf{H}_n^{-1}(a_0,p_0)\cdot\mathbf{G}_n(a_0,p_0) \xrightarrow[n\to\infty]{L} \begin{pmatrix} 1 \\ -1 \end{pmatrix} Z. \tag{E.8}$$

In fact, since $(Z_k)_k$ is a i.i.d. sequence of r.v. with mean 2 and variance 4 (see (E.1)), we get,

$$E(G_n^{(1)}(a_0,p_0)G_n^{(2)}(a_0,p_0)) = -E\Big(\frac{1}{a_0}\sum_{k=1}^n \log(k)(2-Z_k)^2\Big) = -\frac{4}{a_0}\sum_{k=1}^n \log(k)$$

and hence

$$Corr\Big(\frac{a_0}{2\sqrt{n}}G_n^{(1)}(a_0,p_0), \frac{G_n^{(2)}(a_0,p_0)}{2\sqrt{\sum_{k=1}^n \log^2(k)}}\Big) = \frac{-\sum_{k=1}^n \log(k)}{\sqrt{n}\sqrt{\sum_{k=1}^n \log^2(k)}} \xrightarrow[n\to\infty]{} -1.$$

Now, since

$$\dot{\mathbf{G}}(a_0,p_0) = \begin{pmatrix} -\frac{4n}{a_0^2}\Big(1+\frac{3}{4}\frac{\sum_{k=1}^n (Z_k-2)}{n}\Big) & \frac{4\sum_{k=1}^n \log(k)}{a_0}\Big(1+\frac{\sum_{k=1}^n \log(k)(Z_k-2)}{2\sum_{k=1}^n \log(k)}\Big) \\ \frac{4\sum_{k=1}^n \log(k)}{a_0}\Big(1+\frac{\sum_{k=1}^n \log(k)(Z_k-2)}{2\sum_{k=1}^n \log(k)}\Big) & -4\sum_{k=1}^n \log^2(k)\Big(1+\frac{\sum_{k=1}^n \log^2(k)(Z_k-2)}{2\sum_{k=1}^n \log^2(k)}\Big) \end{pmatrix}$$
$$\asymp \begin{pmatrix} -\frac{4n}{a_0^2} & \frac{4\sum_{k=1}^n \log(k)}{a_0} \\ \frac{4\sum_{k=1}^n \log(k)}{a_0} & -4\sum_{k=1}^n \log^2(k) \end{pmatrix}$$

then, by (E.8) (see [6, pag.191]), we get

$$\mathbf{H}_n^{-1}(a_0,p_0)\cdot(-\dot{\mathbf{G}}(a_0,p_0))\cdot\begin{pmatrix}\hat{a}_n-a_0\\ \hat{p}_n-p_0\end{pmatrix} \xrightarrow[n\to\infty]{L} \begin{pmatrix}1\\-1\end{pmatrix} Z,$$

which is the thesis, once the conditions of uniformly boundedness are checked as for the previous case. □

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